Test of semiclassical amplitudes for quantum ray-splitting systems

A. Kohler*
Sysdeco GIS AS, P.O. Box 433, 3604 Kongsberg, Norway

R. Blümel†
Department of Physics, Wesleyan University, Middletown, Connecticut 06459-0155
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We compute semiclassically and numerically the weights of ray-splitting orbits in the density of states of a rectangular and an annular ray-splitting billiard. The agreement between the semiclassical and the numerical results is very good, confirming the necessity of including reflection and transmission coefficients of non-Newtonian ray-splitting orbits in semiclassical expressions for the density of states of ray-splitting systems. [S1063-651X(99)02206-0]

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Ray splitting occurs in all wave systems where the wavelength is large in comparison with the range over which the potential changes. Thus ray splitting is a phenomenon that is important in many fields of physics. Examples are readily found: In optics ray splitting occurs at the interface between two dielectrica of different indices of refraction. In acoustics ray splitting occurs, e.g., when a wave generated by an earthquake hits a fault line. The investigation of ray splitting was initiated by Couchman *et al.* [1], who studied ray splitting in the field of acoustics and quantum chaos. An important aspect in quantum ray-splitting systems is that the underlying classical mechanics is non-Newtonian and nondeterministic with above-barrier reflection [1–6]. Experimental evidence for the signatures of non-Newtonian orbits in ray-splitting systems was given by Sirko *et al.* [7] and Bauch *et al.* [8]. Sirko *et al.* [7] identified the signatures of non-Newtonian orbits in the spectrum of a Teflon-loaded microwave cavity. Bauch *et al.* [8] amplified the results of [7] and in addition investigated ray splitting in a metal-loaded microwave cavity. Modifications of Gutzwiller's trace formula [1] and Bogomolny's transfer operator [2] have been suggested in order to accommodate ray splitting in a semiclassical context. The focus of this paper is a numerical test of the modified Bogomolny transfer operator for ray-splitting systems [2] given by

$$T(q,q',E)=\sum_j \frac{A_j}{\sqrt{2\pi i\hbar}}\left[\left|\frac{\partial S_j(q,q',E)}{\partial q\,\partial q'}\right|\right]^{1/2} \times e^{(i/\hbar)S_j(q,q',E)+i\delta_j}, \tag{1}$$

where the summation is over all Newtonian and non-Newtonian orbits j of energy E with starting point q' and end-point q in the Poincaré surface of section, $S_j(q,q',E)$ is the classical action of orbit j, δ_j is its phase and [1]

$$A_j=\left[\prod_{i=1}^{\varrho_j}|r_{ij}|^2\right]\left[\prod_{k=1}^{\tau_j}(1-|r_{kj}|^2)\right], \tag{2}$$

where ϱ_j (τ_j) are the number of reflections (transmissions) encountered by orbit j, and r_{ij} is the reflection coefficient at the ith reflection. The advantage of the Bogomolny transfer operator is that it can be used for stable, marginally stable, and unstable orbits in the same way. In terms of the Bogomolny transfer operator the fluctuating part of the density of states reads [2]

$$\tilde{\rho}(E)=\frac{1}{\pi}\Im\sum_{s=1}^{\infty}\frac{1}{s}\frac{d}{dE}\mathrm{Tr}\,T(E)^s=-\frac{1}{\pi}\Im\sum_{s=1}^{\infty}\tilde{\rho}_s(E). \tag{3}$$

It is well known that the Fourier transform of the density of states,

$$F(l)=\int\tilde{\rho}(E)\exp(-il\sqrt{E})dE, \tag{4}$$

exhibits peaks at the classical actions of classical periodic orbits [9]. This means that Fourier-transforming (3) and the quantum spectrum of a ray-splitting billiard analytically and numerically allows us to test the weights A_j of ray-splitting orbits. We performed this test focusing on the simplest non-Newtonian ray-splitting orbits of two ray-splitting billiards, a rectangular and an annular ray-splitting billiard. Dirichlet boundary conditions are used in both cases.

The rectangular ray-splitting billiard is shown in Fig. 1. It consists of two parts, one held at potential $V=0$, the other at potential $V=V_0$. The reflection coefficient of the dashed-line orbit in Fig. 1, reflected off the ray-splitting boundary separating the two potential regions, is given by

$$r=\frac{\sqrt{E}\cos\beta-\sqrt{E-V_0}\cos\beta'}{\sqrt{E}\cos\beta+\sqrt{E-V_0}\cos\beta'}, \tag{5}$$

where β is the angle of incidence and

$$\sin(\beta)=\sqrt{1-V_0/E}\sin(\beta'). \tag{6}$$

Since r is energy dependent, Fourier-transforming $\tilde{\rho}(E)$ according to Eq. (4) means that the coefficients A_j, too, are affected by the integration over the energy E. This way, therefore, it is not possible to determine A_j correctly. The use

*Electronic address: Achim.Kohler@gis.sysdeco.no
†Electronic address: rblumel@wesleyan.edu

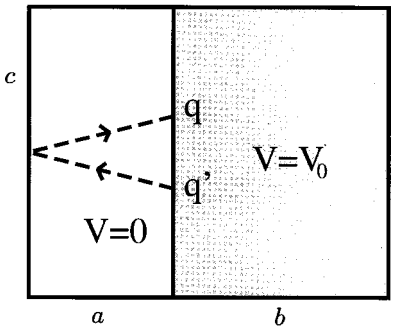

FIG. 1. Rectangular ray-splitting billiard with a step potential. It consists of a rectangular domain divided into two rectangular subdomains held at constant potentials $V=0$ and $V=V_0$, respectively. An orbit originating at point q' and terminating at point q is also shown.

of scaled spectroscopy, i.e., to calculate and Fourier-transform quantal spectra with constant ratio $\eta=V_0/E$, solves this problem. This can be seen explicitly if we rewrite Eq. (5) using $\eta=V_0/E$:

$$r=\frac{\cos\beta-\kappa\cos\beta'}{\cos\beta+\kappa\cos\beta'}, \tag{7}$$

where $\kappa=\sqrt{1-\eta}$. Performing the trace over T, only the periodic orbits starting at q and ending at $q'=q$ survive. Their contribution to Eq. (3) is given by

$$\tilde{\rho}_1(E)=\frac{c|r|}{\sqrt{a\pi}2\pi k^{3/2}}\left[\frac{1}{4}\sin(k\,l_1+3\pi/4)+\frac{1}{2}kl_1\cos(kl_1+3\pi/4)\right], \tag{8}$$

where a and c are dimensions of the billiard (see Fig. 1), $k=\sqrt{E}$, $l_1=2a$, and

$$r=\frac{1-\kappa}{1+\kappa}. \tag{9}$$

The dashed line in Fig. 2 shows the Fourier transform of Eq. (8). The smooth line in Fig. 2 is the Fourier transform of $\tilde{\rho}(E)$ of the rectangular ray-splitting billiard computed on the basis of the first 20 000 energy levels for $a=1$, $b=8$, $c=8$, and $\eta=0.5$. As expected, both Fourier transforms peak at $l_1=2a=2$. Although a small deviation between the analytical and the numerical results is visible, Fig. 2 shows convincingly that the reflection probability $r\approx0.1716$ computed from Eq. (9) is semiclassically valid. Further investigation showed that the deviation in Fig. 2 depends on the number of energy levels included in the calculations. We found numerically that to a very good accuracy the relative deviation of the analytical result from the numerical result tends to zero according to $1/\sqrt{k_{\max}}$, where $E_{\max}=k_{\max}^2$ is the largest energy level included in the Fourier transform. Thus the analytical and the numerical results agree in the limit $k_{\max}\to\infty$.

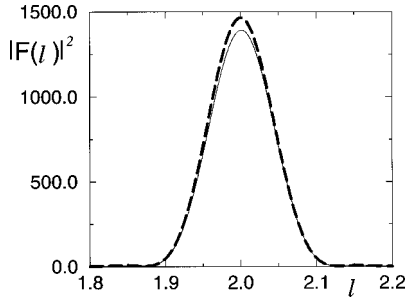

FIG. 2. Fourier transform of the fluctuating part of the level density $\tilde{\rho}(E)$ for the rectangular ray-splitting billiard. The dashed line is the analytical result. The full line is the Fourier transform of the numerically obtained quantum spectrum.

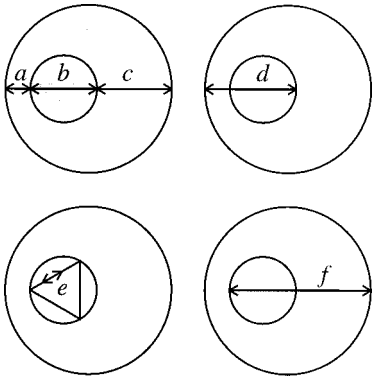

FIG. 3. The annular ray-splitting billiard together with some short non-Newtonian orbits. The shaded areas are held at potential $V=V_0$, the white areas are at zero potential.

The annular ray-splitting billiard [10] consists of two nested circular domains, the inner one at potential $V=V_0$, the outer one at potential $V=0$ (see Fig. 3). In order to determine peak heights we calculated 800 energy values with both positive and negative parity for the scaled problem with $\eta=0.5$ and obtained the level density $\tilde{\rho}(E)$. The Fourier transform of $\tilde{\rho}(E)$ in the range $0<l<2$ is shown as the full line in Fig. 4. The letters labeling the peaks refer to the orbits shown in Fig. 3. In the range $0<l<2$ only non-Newtonian orbits exist. The orbit labeled ''f'' has optical path length $l\approx3.014$, outside of the chosen l range.

We focus now on the periodic orbit labeled ''b'' in Fig. 3.

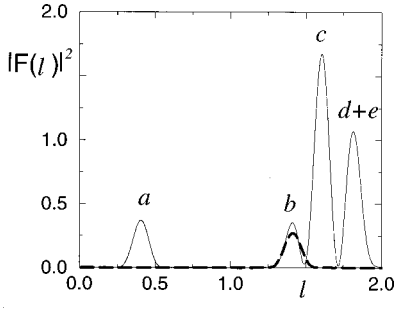

FIG. 4. Fourier transform of 800 energy levels of the annular ray-splitting billiard, including levels with both positive and negative parity (full line). The letters at the peaks refer to the orbits in Fig. 3. For $0<l<2$ only non-Newtonian orbits occur. The dashed line corresponds to the Fourier transform of the density of states calculated analytically with the Bogomolny transfer operator approach taking only the orbit ''b'' of Fig. 3 into account.

Its contribution to the fluctuating part of the density of states is given by

$$\tilde{\rho}_2(E)=\frac{a\sqrt{l_2}|r|^2}{\sqrt{2\pi}2k^{3/2}}\left[\frac{1}{4}\sin(kl_2+3\pi/4)+\frac{1}{2}kl_2\cos(kl_2+3\pi/4)\right], \quad (10)$$

where a is the radius of the inner circle and $l_2=4a\kappa$. Because orbit ''b'' hits the ray-splitting boundary twice, its weight is $|r|^2\approx 0.029\,44$, where $r=(\kappa-1)/(\kappa+1)$. The Fourier transform of Eq. (10) is shown as the dashed line in Fig. 4. The agreement between the analytically and numerically computed peak heights is very good. The deviation of the analytical result from the numerical result is again explained by the finite number of energy levels included in the calculations.

Summarizing, we calculated analytically the weights of simple non-Newtonian ray-splitting orbits in the Fourier transform of the spectra of two different ray-splitting billiards and compared the analytical results with the weights obtained from the Fourier transform of numerically computed spectra. The analytically and the numerically calculated weights are in excellent agreement. Thus, within the scope of our checks, the ray-splitting modification of Bogomolny's transfer operator proposed in [2] is semiclassically valid.

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